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Pedagogy in Action: Complex Numbers with TI-Nspire and Visible Learning

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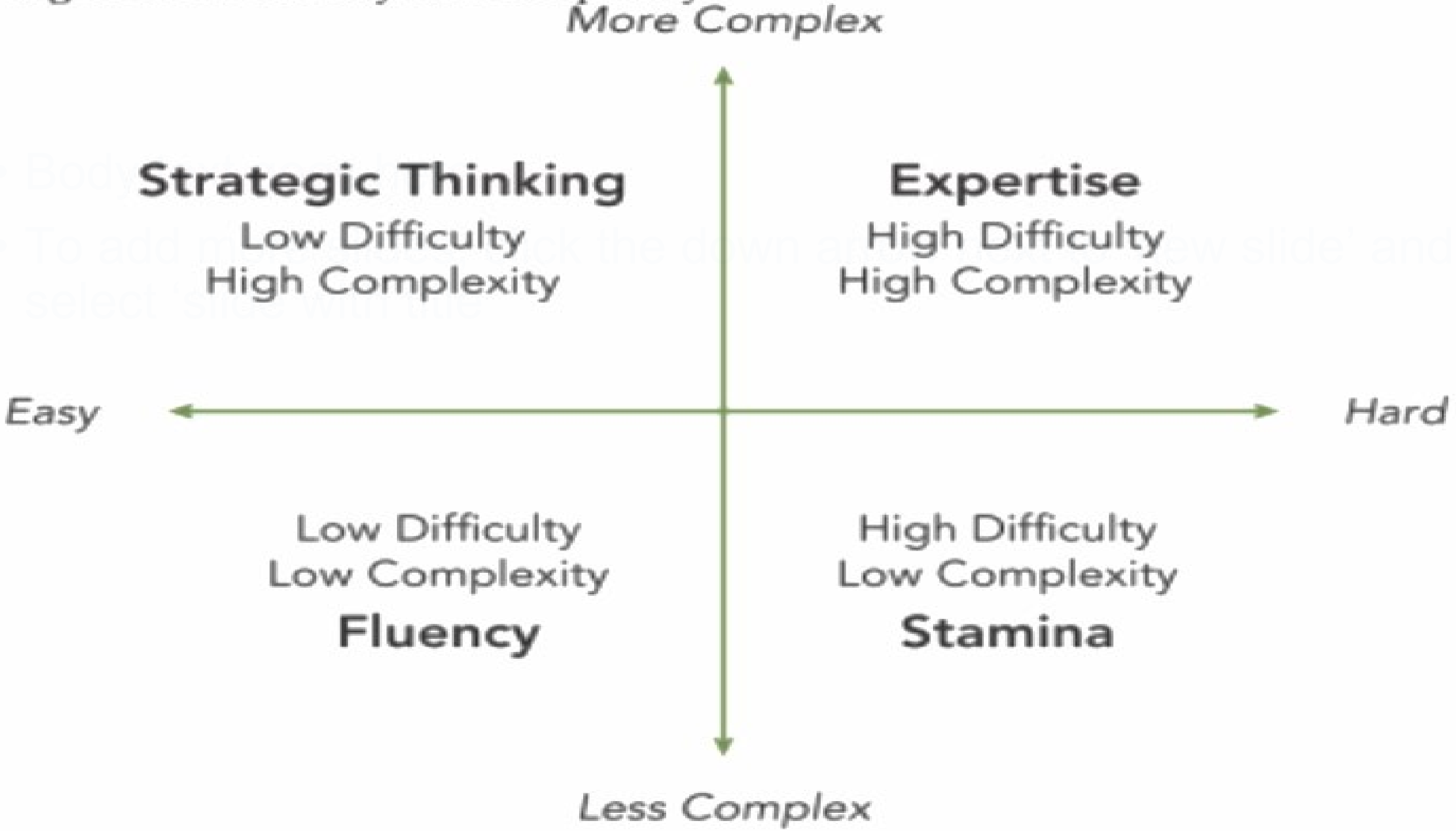
Imaginary friends

- Introducing Complex Numbers to students for the first time

    - Arithmetic or geometric?

Where automacity resides.

Figure 3.1 Difficulty and Complexity



Quadrant Walk…

Fluency (Low Difficulty, Low Complexity)

- **Definition:** This quadrant is about mastering the foundational operations and representations.
- **Example Activity:** Plotting complex numbers on the Argand plane or performing basic operations like addition and subtraction.
- **Goal:** Build automacity by practicing until students no longer need to think about the mechanics.
- **Visual Aid:** Show students repeatedly plotting simple numbers like $3+4i$ or $2-i$, and describe how this builds a visual intuition.

Quadrant Walk…

Stamina (High Difficulty, Low Complexity)

- **Definition:** Tasks that challenge students' persistence but remain conceptually straightforward.
- **Example Activity:** Compute the modulus and argument for increasingly difficult numbers or powers of i.
- **Goal:** Encourage sustained effort and resilience. Students need to stick with the task even when the calculations become lengthy.
- **Visual Aid:** Display how i^n cycles through results and grows challenging for higher n.

Quadrant Walk…

Strategic Thinking (Low Difficulty, High Complexity)

- **Definition:** Here, students engage with conceptually rich problems that require higher-order thinking but involve familiar operations.
- **Example Activity:** Explore how multiplying two complex numbers represents a geometric transformation (scaling and rotation).
- **Goal:** Help students see connections, e.g., link Euler's formula $e^{i\theta} = cos\theta + isin\theta$ to the polar form of complex numbers.
- **Visual Aid:** Use graphing tools (e.g., Desmos or TI-Nspire) to demonstrate rotations and dilations dynamically on the Argand plane.

Quadrant Walk…

Expertise (High Difficulty, High Complexity)

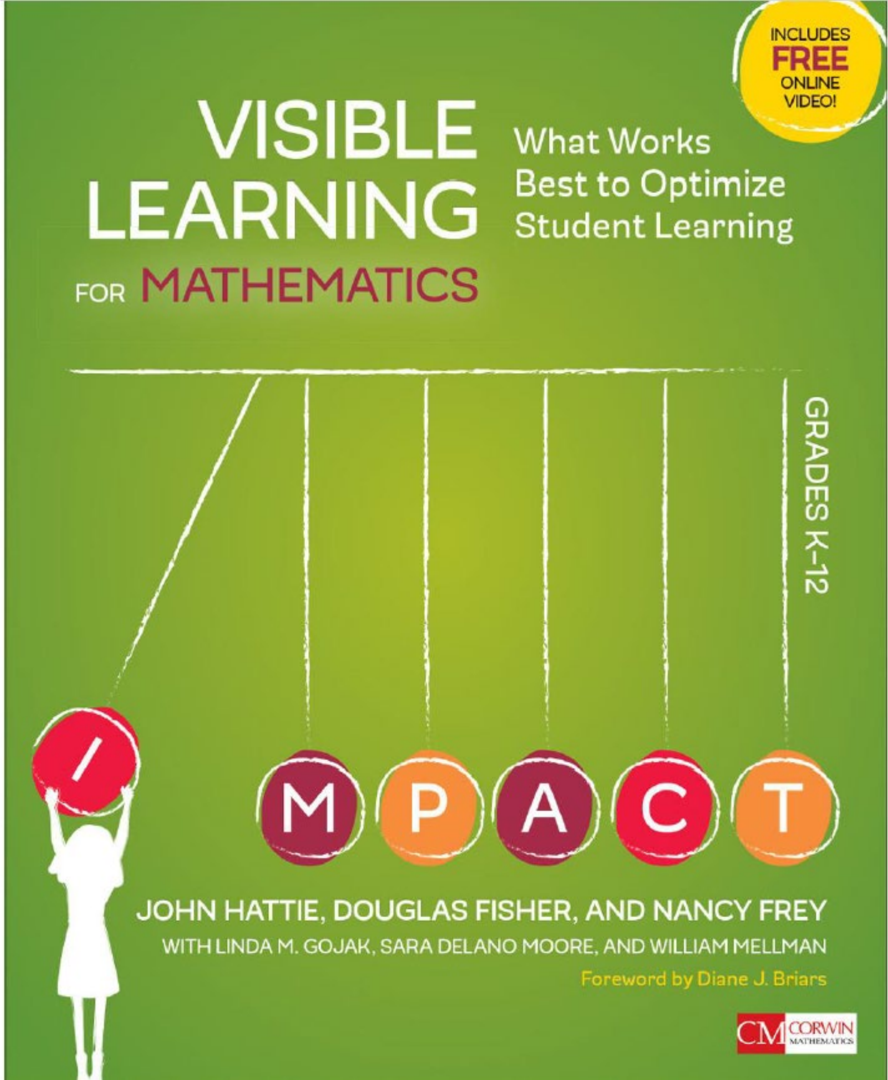
- **Definition:** Advanced, multi-layered problems that combine rigorous computation and deep conceptual thinking.
- **Example Activity:** Investigate the Mandelbrot set or prove identities using Euler's formula.
- **Goal:** Encourage students to synthesize knowledge and apply it creatively.
- **Visual Aid:** Show a zoomed-in Mandelbrot set visualization and explain its connection to iterative processes in complex numbers.

Visible learning for Mathematics

4.7 out of 5

339 global ratings

5 star	83%
4 star	10%
3 star	4%
2 star	1%
1 star	2%



Surface Learning
• Begins with the development of conceptual understanding, then labels (vocabulary) and procedures are explicitly introduced to give structure to concepts
Deep Learning
• About noticing relationships, extending ideas to new situations, and making connections to new ideas or representations • Students move to deep learning when they plan, investigate, and elaborate on their conceptual understandings, and then begin to make generalizations
Transfer Learning

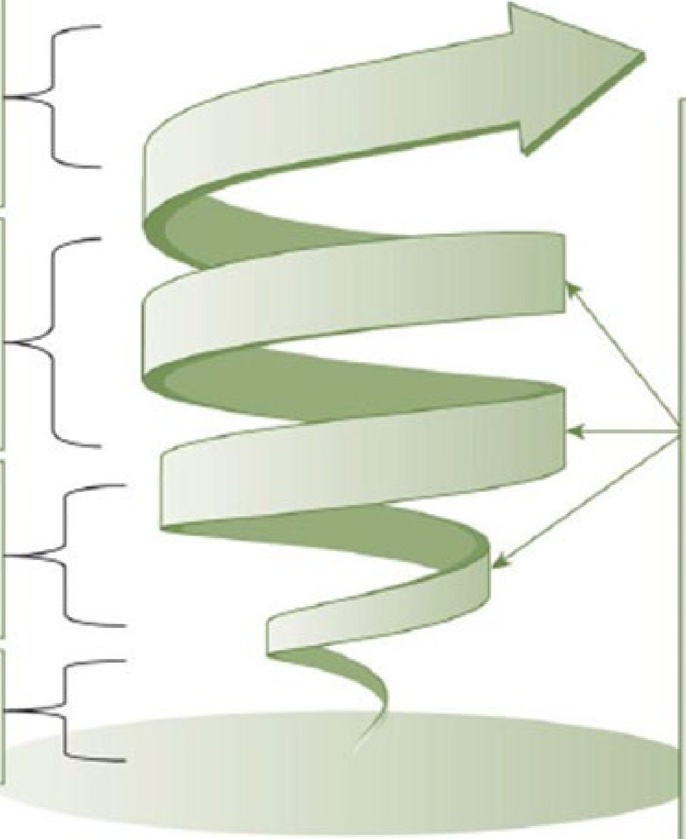
• Students take the reins of their own learning and are able to apply their thinking to new contexts and situations
  • Transfer by detecting similarities and differences between concepts and situations
  • Close association between a previously learned task and a novel situation is necessary for promoting transfer of learning

Transfer: Apply conceptual understanding and skills—with little teacher assistance—to new and parallel contexts and scenarios and future units of study

Deep: Deepen understanding by making conceptual connections between and among concepts and applying and practicing procedural skills

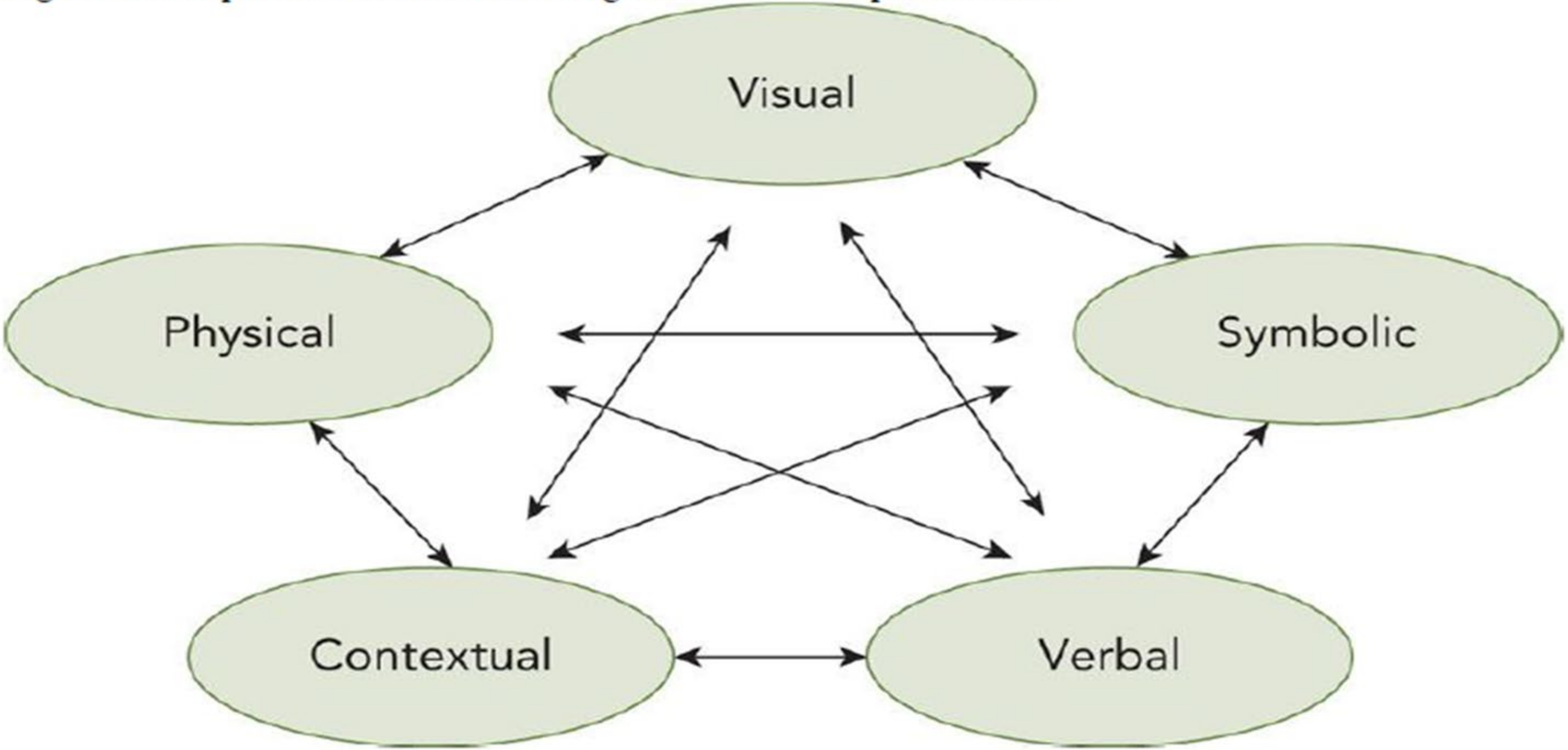
Surface: Build initial understanding of concepts, skills, and vocabulary on a new topic

Leverage prior knowledge from previous unit



In any given unit of study, your ongoing, continuous assessment will tell you that your learners are in various places in their learning along this path, and will sometimes move back and forth between surface and deep as they build understanding. Transfer happens when students apply what they know to new situations or new concepts. It is your goal to provide the interventions and strategies they need at the right time for the right reason.

Figure 5.12 Important Connections Among Mathematical Representations



Source: National Council of Teachers of Mathematics (2014, p. 25).

Using Multiple Representations to Promote Deep Learning

An investigative approach to complex numbers

1. This question asks you to explore cubic polynomials of the form $(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ and corresponding cubic equations with one real root and two complex roots of the form $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$.

In parts (a), (b) and (c), let $r=1$, $a=4$ and $b=1$.

Consider the equation $(z-1)(z^2-8z+17)=0$ for $z \in \mathbb{C}$.

(a) (i) Given that 1 and $4+\mathrm{i}$ are roots of the equation, write down the third root.

(ii) Verify that the mean of the two complex roots is 4.

Consider the function $f(x)=(x-1)(x^2-8x+17)$ for $x \in \mathbb{R}$.

(b) Show that the line $y=x-1$ is tangent to the curve $y=f(x)$ at the point $\mathrm{A}(4, 3)$.

(c) Sketch the curve $y=f(x)$ and the tangent to the curve at point A, clearly showing where the tangent crosses the x-axis.

Consider the function $g(x)=(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

(d) (i) Show that $g'(x)=2(x-r)(x-a)+x^2-2ax+a^2+b^2$.

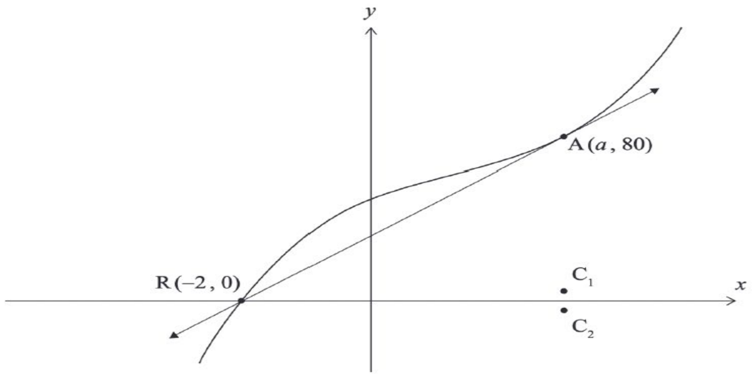
(ii) Hence, or otherwise, prove that the tangent to the curve $y=g(x)$ at the point $\mathrm{A}(a, g(a))$ intersects the x-axis at the point $\mathrm{R}(r, 0)$.

The equation $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$ has roots r and $a \pm b\mathrm{i}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

(e) Deduce from part (d)(i) that the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$ can be expressed as $a \pm \mathrm{i}\sqrt{g'(a)}$.

On the Cartesian plane, the points $\mathrm{C}_1\left(a, \sqrt{g'(a)}\right)$ and $\mathrm{C}_2\left(a, -\sqrt{g'(a)}\right)$ represent the real and imaginary parts of the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$.

The following diagram shows a particular curve of the form $y=(x-r)(x^2-2ax+a^2+16)$ and the tangent to the curve at the point $\mathrm{A}(a, 80)$. The curve and the tangent both intersect the x-axis at the point $\mathrm{R}(-2, 0)$. The points C_1 and C_2 are also shown.



(f) (i) Use this diagram to determine the roots of the corresponding equation of the form $(z-r)(z^2-2az+a^2+16)=0$ for $z \in \mathbb{C}$.

(ii) State the coordinates of C_2.

Consider the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a \neq r$, $b>0$. The points $\mathrm{A}(a, g(a))$ and $\mathrm{R}(r, 0)$ are as defined in part (d)(ii). The curve has a point of inflexion at point P.

(g) (i) Show that the x-coordinate of P is $\frac{1}{3}(2a+r)$.

You are **not** required to demonstrate a change in concavity.

(ii) Hence describe numerically the horizontal position of point P relative to the horizontal positions of the points R and A.

Consider the special case where $a=r$ and $b>0$.

(h) (i) Sketch the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a=r=1$ and $b=2$.

(ii) For $a=r$ and $b>0$, state in terms of r, the coordinates of points P and A.

(a) (i) $4-\mathrm{i}$

(ii) mean $=\frac{1}{2}(4+\mathrm{i}+4-\mathrm{i})$

$=4$

(b)

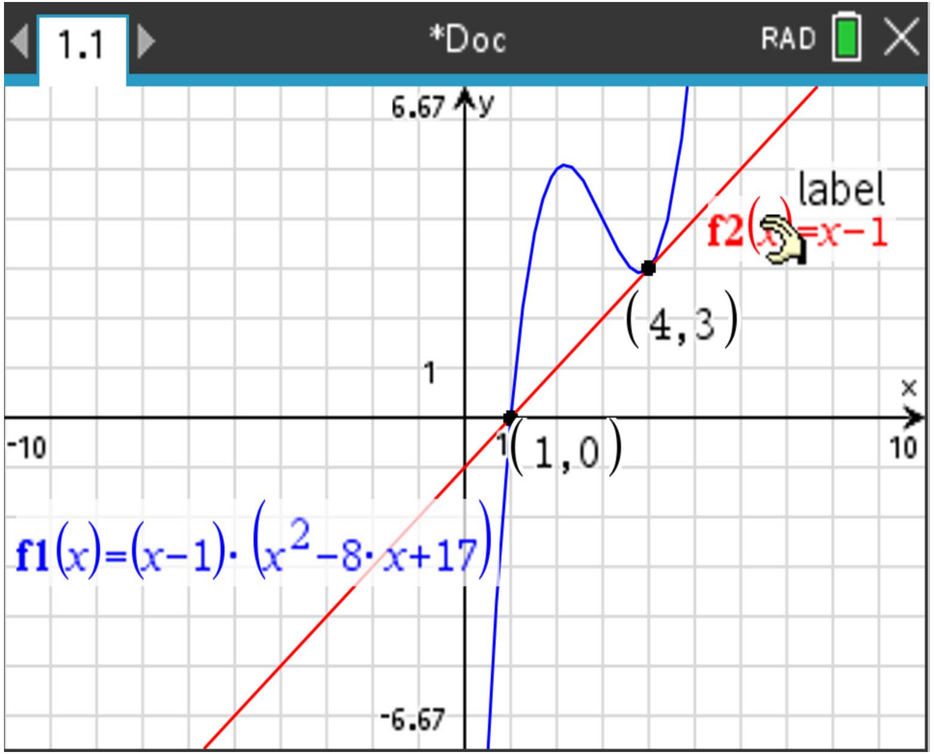
$f'(x)=(x-1)(2x-8)+x^2-8x+17 \ \left(f'(x)=3x^2-18x+25\right)$

$f'(4)=1$

$y-3=1(x-4)$

so $y=x-1$ is the tangent to the curve at $\mathrm{A}(4,\ 3)$

(c)



(d) (i) **EITHER**

$$g'(x)=(x-r)(2x-2a)+x^2-2ax+a^2+b^2$$

OR

$$g(x)=x^3-(2a+r)x^2+(a^2+b^2+2ar)x-(a^2+b^2)r$$

attempts to find $g'(x)$

$$g'(x)=3x^2-2(2a+r)x+a^2+b^2+2ar$$

$$=2x^2-2(a+r)x+2ar+x^2-2ax+a^2+b^2$$

$$\left(=2(x^2-ax-rx+ar)+x^2-2ax+a^2+b^2\right)$$

THEN

$$g'(x)=2(x-r)(x-a)+x^2-2ax+a^2+b^2$$

a positive cubic with an x- intercept $(x=1)$, and a local maximum and local minimum in the first quadrant both positioned to the left of A

$g(a)=b^2(a-r)$

$g'(a)=b^2$

attempts to substitute their $g(a)$ and $g'(a)$ into $y-g(a)=g'(a)(x-a)$

$y-b^2(a-r)=b^2(x-a)$

EITHER

$y=b^2(x-r)$ $\left(y=b^2x-b^2r\right)$

sets $y=0$ so $b^2(x-r)=0$

$b>0 \Rightarrow x=r$ OR $b\neq 0 \Rightarrow x=r$

OR

sets $y=0$ so $-b^2(a-r)=b^2(x-a)$

$b>0$ OR $b\neq 0$ $\Rightarrow -(a-r)=x-a$

$x=r$

THEN

so the tangent intersects the x- axis at the point $\mathrm{R}(r, 0)$

(e) **EITHER**

$g'(a) = b^2 \Rightarrow b = \sqrt{g'(a)}$ (since $b > 0$)

$(a \pm b\mathrm{i} =) a \pm \mathrm{i}\sqrt{b^2}$ and $g'(a) = b^2$

THEN

hence the complex roots can be expressed as $a \pm \mathrm{i}\sqrt{g'(a)}$

(f) (i) $b = 4$ (seen anywhere)

EITHER

attempts to find the gradient of the tangent in terms of a and equates to 16

OR

substitutes $r = -2, x = a$ and $y = 80$ to form $80 = (a - (-2))(a^2 - 2a^2 + a^2 + 16)$

OR

substitutes $r = -2, x = a$ and $y = 80$ into $y = 16(x - r)$

THEN

$\frac{80}{a+2} = 16 \Rightarrow a = 3$

roots are -2 (seen anywhere) and $3 \pm 4\mathrm{i}$

(ii) $(3, -4)$

(g) (i) $g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$

attempts to find $g''(x)$

$g''(x) = 2(x-a) + 2(x-r) + 2x - 2a \ \left(= 6x - 2r - 4a\right)$

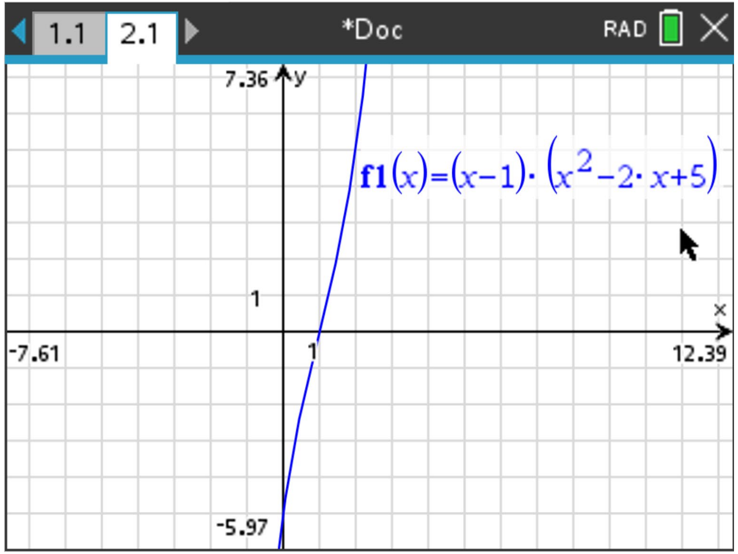
sets $g''(x) = 0$ and correctly solves for x

for example, obtaining $x - r + 2(x-a) = 0$ leading to $3x = 2a + r$

so $x = \frac{1}{3}(2a + r)$

(ii) point P is $\frac{2}{3}$ of the horizontal distance (way) from point R to point A

(h) (i) $y=(x-1)(x^2-2x+5)$



a positive cubic with no stationary points and a non-stationary point of inflexion at $x=1$

(ii) $(r\,,\,0)$

Think, pair, share

Here is a 'surface learning' task

Can you modify or build upon it to incorporate 'deep learning'?

(there is no one right answer)

Consider the equations

$$z^2 - 1 = 0$$

$$z^3 - 1 = 0$$

$$z^4 - 1 = 0$$

a. Find the solutions to each of the equations in polar form and plot the solutions on separate Argand diagrams
b. What do the solutions of the equations form on the Argand diagrams?
c. Predict the shape that the solutions to $z^5 - 1 = 0$ would form.

One possible suggestion

- What do solutions of the equation $z^n - 1 = 0, n \geq 3$ form on an Argand diagram?

- Use the geometrical representation of the solutions to $z^n - 1 = 0, n \geq 3$ predict the polar form of the solutions to the equation

No Plane No Gain

- Let's take 1-2 questions (from our compiled questions) that students (traditionally) struggle with.

- How can we adapt the question to introduce: transfer learning (i.e. connect representations & metacognition) to enhance learning?
    - Take a question that has multiple methods and focus on metacognition

Have teachers share their thoughts.

-

Pedagogical takeaways…

- **Progression is Key:** Move students through the quadrants intentionally, with automacity as the cornerstone.
- **Visible Learning:** Use graphing tools to make abstract concepts concrete, especially for Strategic Thinking and Expertise.
- **Differentiation:** Provide tasks tailored to each quadrant to address diverse learner needs.
- **Engagement:** Incorporate real-world applications, like fractals or signal processing, to demonstrate the relevance of complex numbers.

Work Cited

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